Applications of Topological Holographic Principle

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CUHK

Based on works with :
Kaixin Ji, Ce Shen, Yidun Wan, Yu Zhao 2506.05324 (Accepted in principle, Nature Physics)
Gong Cheng, Lin Chen, Zheng-Cheng Gu 2311.18005 Phys.Rev.X 15 (2025) 1, 011073
Lin Chen, Kaixin Ji, Haochen Zhang, Ce Shen, Ruoshui Wang
2210.12127 Physical Review X 14 (4), 041033
Hao Geng, Yikun Jiang 2505.20385
Yikun Jiang, Bingxin Lao, 2504.21660
Lin Chen, Yikun Jiang, Bingxin Lao
2404.00877, 2403.03179 *SciPost Phys.* 19 (2025) 6, 163

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Generalised Symmetries

- Generalised symmetry — rather than focusing on group actions, focus on topological operators. If topological operators are of higher co-dimensions, they correspond to “higher symmetries”

- e.g. in usual group symmetry the charge operators are topological co-dimension 1 operators. 1-form symmetry -> co-dimension 2 operators.

- The collection of these operators form a mathematical structure: a (higher) fusion category — we will need only 1-(spherical) fusion category in the following to discuss 1d topological defects of 2D CFT.

Generalized symmetries:
Gaiotto, Kapustin, Seiberg, Willett 2014;
Non-invertible symmetries:
Verlinde 1988;
Moore, Seiberg 1988-1989;
Petkova, Zuber 2000,
(Frohlich), Fuchs, Runkel, Schweigert 2002-2006;
Bhardwaj, Tachikawa 2017;
Chang, Lin, Shao, Wang, Yin 2018;
Kong, Zheng 2019;
Ji, Wen 2019;
Thorngren, Wang 2019;

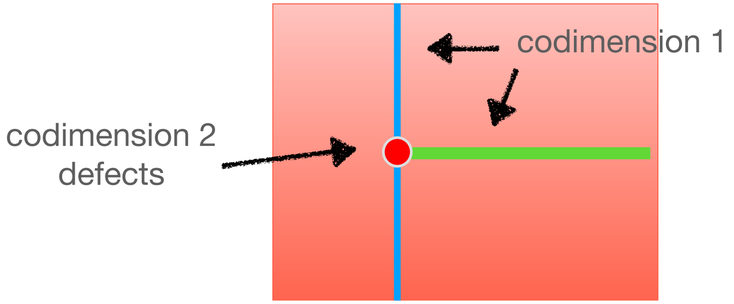
Sandwich and SymTFT:
Kong, Wen, Zheng 2015, 2017;
Kong, Zheng 2019;
Ji, Wen 2019;
Kong, Lan, Wen, Zhang, Zheng 2020;
Gaiotto, Kulp 2020;
Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki 2021;
Freed, Moore, Teleman 2022;

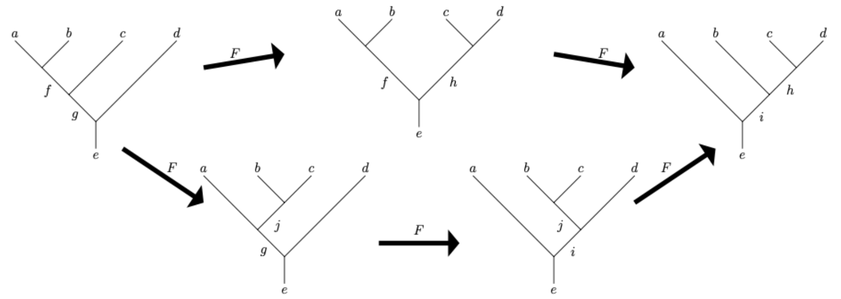
Reviews:
Cordova, Dumitrescu, Intriligator, Shao 2022;
Schafer-Nameki 2023; Shao 2023;
McGreevy 2022

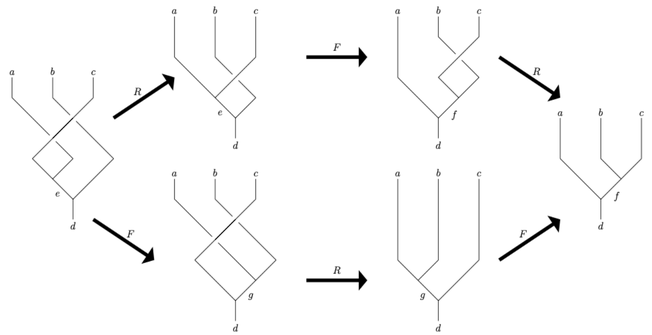
Generalised Symmetries

Bhardwaj, Tachikawa 2017;
Chang, Lin, Shao, Wang, Yin 2019;
Thorngren, Wang 2019; Ji, Wen 2020;
Kong, Lan, WenZhang, Zheng, 2020 ……

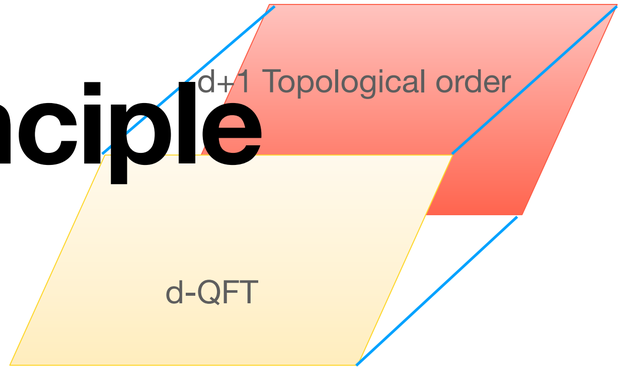
- codimension 1 operators — objects
- codimension 2 operators — 1 morphisms between objects etc
- These operators can fuse and the fusion should satisfy some consistency conditions e.g. associativity constraints
  e.g. Pentagon equation for 1-fusion category
  (where 2-morphisms are trivial)
- e.g. 2d CFT topological
  lines called the Verlinde lines have been studied in detail - they form a fusion category







Topological Holographic Principle



- Topological Holographic Principle states that a symmetric d-dimensional QFT can be expressed as a d+1 dimensional TQFT
  path-integral over a “sandwich”
- One boundary is painted a topological boundary condition
- The other is a “dynamical boundary condition”
- The bulk gauges the global symmetries and impose them explicitly, and it is usually called the SymTFT/SymTO.
- Topological defects in the d-QFT corresponds to topological excitations (e.g. anyon lines) in the SymTO.
- e.g. Verlinde lines in 2D CFT ⟶ Wilson lines/anyon lines in a corresponding 3D Chern-Simons theory.

Sandwich and SymTFT:
Kong, Wen, Zheng 2015, 2017;
Kong, Zheng 2019;
Ji, Wen 2019;
Kong, Lan, Wen, Zhang, Zheng 2020;
Gaiotto, Kulp 2020;
Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki 2021;
Freed, Moore, Teleman 2022;

The Strange correlator is an explicit realisation.

Overview

- Constructing Critical models - predicting critical couplings and phase boundaries
- symmetry preserving TRG algorithms
- Discretising CFTs
- Holographic tensor network in 2D

Classical Statistical Lattice models

Historically a treasure of CFTs

- Statistical lattice models are important area of studies both in mathematics and physics. Important mathematical structures and physical phenomena were discovered from studying these models.

- Statistical lattice models played a crucial role in the discovery of many important physical phenomena in phase transitions — notably when it is undergoing continuous/2nd order phase transitions, they have continuous conformal field theory as their thermodynamic limit — this thus produces a UV-complete non-perturbative construction of interacting field theory!

- e.g. The Ising spin model solved by Onsagar — has a critical point which is related to the 2D Ising CFT

- The entire A-series minimal models were originally found in the A-series lattice integrable models.

- This is no longer the main-stream of producing new CFT — writing down Lagrangian has become the main-stream. Why? Can we re-live this good old times?

Turaev Viro TQFT/Levin-Wen model in 2+1 d

Fusion Category

- List of simple objects: {a,b,c,…},
- 1-Morphisms: maps taking a to b, $f \in$ Hom(a,b); $a \xrightarrow{f} b$

  for simple objects: Hom(a,a)={Identity map}, Hom(a,b)=∅ when $a \neq b$.
- Fusion rules: $a \otimes b = \oplus_c N_{ab}^c c$,

  N_{ab}^c: non-negative integers,

  quantum dimension: $d_a \overset{\text{def}}{=}$ max eigenvalue of N_a, $[\ (N_a)_b^c = N_{ab}^c\ ]$

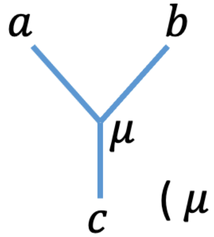
  Associativity of anyon fusion: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$,
- F-symbols

Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

F-symbols

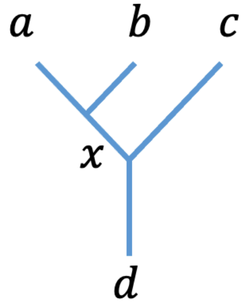
- Fusion diagrams;



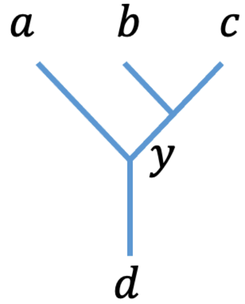
map $a \otimes b$ to c, and form a vector space V_c^{ab}

$(\mu = 1, \dots, N_{ab}^c)$

- Consider the vector space V_d^{abc}, map $a \otimes b \otimes c$ to d,

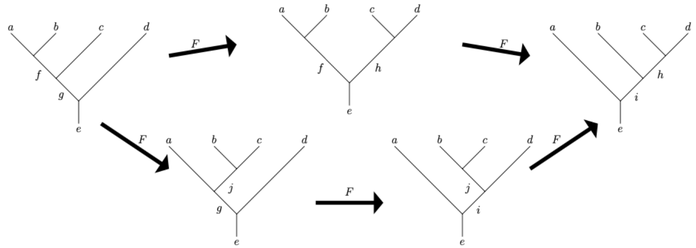


$$=\sum_y F_{d;x,y}^{abc}$$



$(a \otimes b) \otimes c,\ V_d^{abc} \cong \oplus_x V_x^{ab} \otimes V_d^{xc}$

$a \otimes (b \otimes c),\ V_d^{abc} \cong \oplus_y V_d^{ay} \otimes V_y^{bc}$



Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

- A closed oriented 3D manifold $\mathcal{M}$,
- Consider a triangulation $\mathcal{K}$, i.e. split $\mathcal{M}$ into tetrahedrons,
- Give an order of vertices on $\mathcal{K}$, and a fusion category $\mathcal{C}$,
- Assign one object to each edge, one 1-morphism to each triangle, $\mu \in Hom(a \otimes b, c)$, $\mu = 1, \dots, N_{ab}^{c}$. If N_{ab}^{c}=0, this label is not allowed. In the following, consider N_{ab}^{c}=0,1.
- Assign a F symbol (tetrahedral symbol) F_{Δ_3} to each tetrahedron Δ_3 .
- The partition function of Turaev-Viro TQFT:

$$Z_{\mathcal{M},\mathcal{C}} = \frac{1}{|D_{\mathcal{C}}|^{N_v}} \sum_{\{a\} \in Obj(\mathcal{C})} \prod_{a} d_a \prod_{\Delta_3 \in \mathcal{K}} F_{\Delta_3}^{\epsilon(\Delta_3)}$$

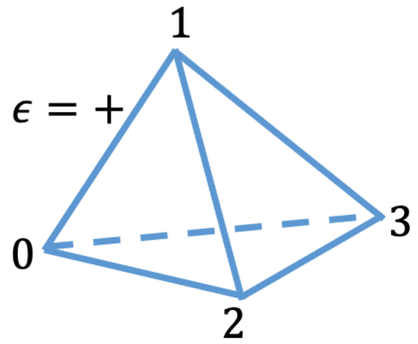
C is called “input category” in the model. The TQFT describes a topological order and the anyons excitations also form a category Z(C) (the center of C) that is called the “output category”

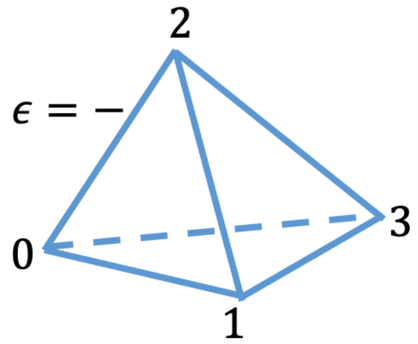
Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

Partition function of TQFT

$$Z_{\mathcal{M},\mathcal{C}} = \frac{1}{|D_{\mathcal{C}}|^{N_v}} \sum_{\{a\}\in Obj(\mathcal{C})} \prod_{a} d_a \prod_{\Delta_3\in\mathcal{K}} F_{\Delta_3}^{\epsilon(\Delta_3)}$$





$D_{\mathcal{C}}$ is the dimension of $\mathcal{C}$.
N_v is the total number of vertices.
$\{a\}$ is a configuration of assigning objects to the edges in $\mathcal{K}$.
$\epsilon(\Delta_3) = \pm 1$ determined by the orientation of the tetrahedron.

The partition function $Z_{\mathcal{M},\mathcal{C}}$ is a topological invariant, i.e. it is independent of the choice of triangulation $\mathcal{K}$ and order of vertices.

Now we will take this TQFT and use it to construct (integrable) lattice models in one lower dimension.

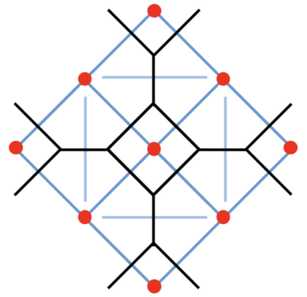
Integrable Lattice Models and Strange Correlators

Path-integral
of a 3-ball with a two dimensional surface

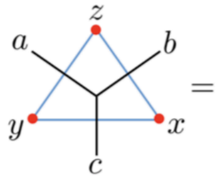


RSOS integrable models and Minimal models and Levin Wen models

Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020;

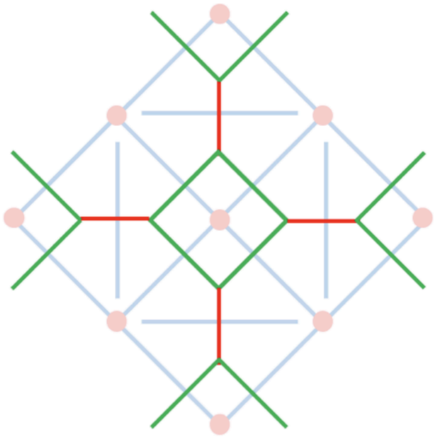


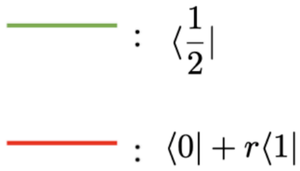
- PEPS representation of Levin-Wen models/Turaev-Viro ground state $|\Psi^{LW}\rangle$
  Gu, Levin, Swingle, Wen PRB 2009; Buershaper, Aguado, Vidal PRB 2009;



$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

- Then pick some mysterious state $\langle \Omega_N |$ and take the overlap with $|\Psi^{LW}\rangle$ i.e. $\langle \Omega_N | \Psi_a^{LW} \rangle$. $\langle \Omega_N |$ is chosen such that the overlap matches exactly the partition function of well known families of integrable models — this is a realisation of the sandwich.





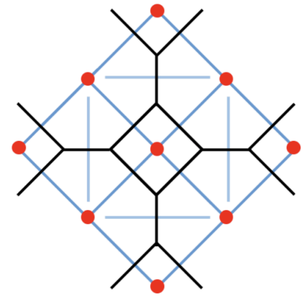
This observation is related to the anyon chain proposed in Feigen et al 2006

RSOS integrable models and Minimal models and Levin Wen models

Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020;

Path-integral
of a 3-ball with a two dimensional surface

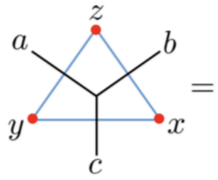




r is related to the temperature in the classical Ising spin model:

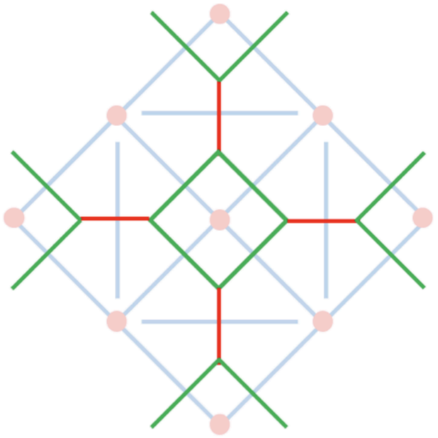
$$r = e^{-2\beta}$$

$$r_c = e^{-2\beta_c}, \qquad \beta_c = 1/2 \ln(1+\sqrt{2})$$



$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

The critical temperature is known for many years. Now expressed as the boundary condition of the Levin-Wen model.



$: \langle \frac{1}{2}|$

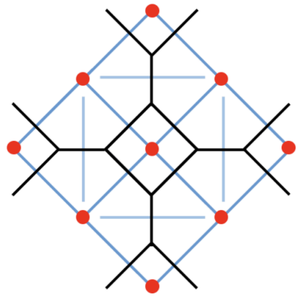
$: \langle 0| + r\langle 1|$

RSOS integrable models and Minimal models and Levin Wen models

Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020; Chen et al 2022;

Path-integral
of a 3-ball with a two dimensional surface

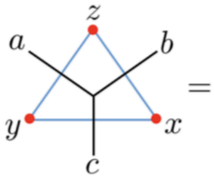




More general there is the A series integrable models with the same ansatz :

Expressed in this form, the well known critical coupling is located at :

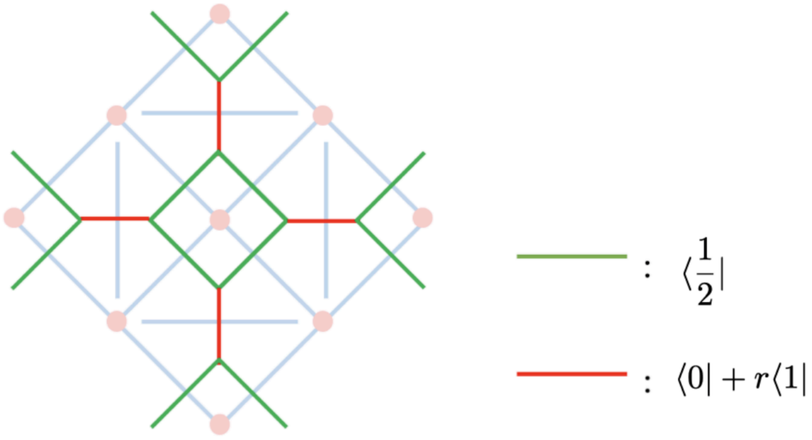
$$r_c = \sqrt{2\cos(2\pi/(k+2))+1}/(2\cos(\pi/(k+2)+1)$$



$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

What is this? Does the Levin Wen/TV model knew about these numbers?

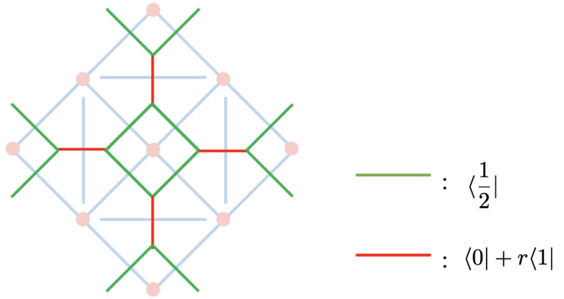
i.e. How exactly is this ansatz chosen in principle????? In particular, which $\langle \Omega_N |$ can produce critical points = CFTs? Is there anything special about these critical couplings???



A 2D CFT Factory : an ansatz of unit cell, and competing anyon condensation

Ansatz of Unit cell

Module and Frobenius algebra



Example: In the Ising example, we notice that at small beta and large beta, the red edges are coloured by one of the two Frobenius algebra of the Ising model respectively. One can check that in this case they are closely related to topological boundary condition of the 3D TV model.

There are 2 Frobenius algebra:

$A_0 = 0$ Electric condensate: equivalent to Lagrangian condensate = $1 \oplus e$

Low temperature Z2 spontaneously sym breaking

How to connect the two things? There are many ways . See Wan et al. But bottom line is to use the Ribbon operator for the Levin Wen model.

$A_1 = 0 \oplus 1$ Magnetic condensate: equivalent to Lagrangian condensate = $1 \oplus m$

High temperature dual Z2 spontaneously sym breaking

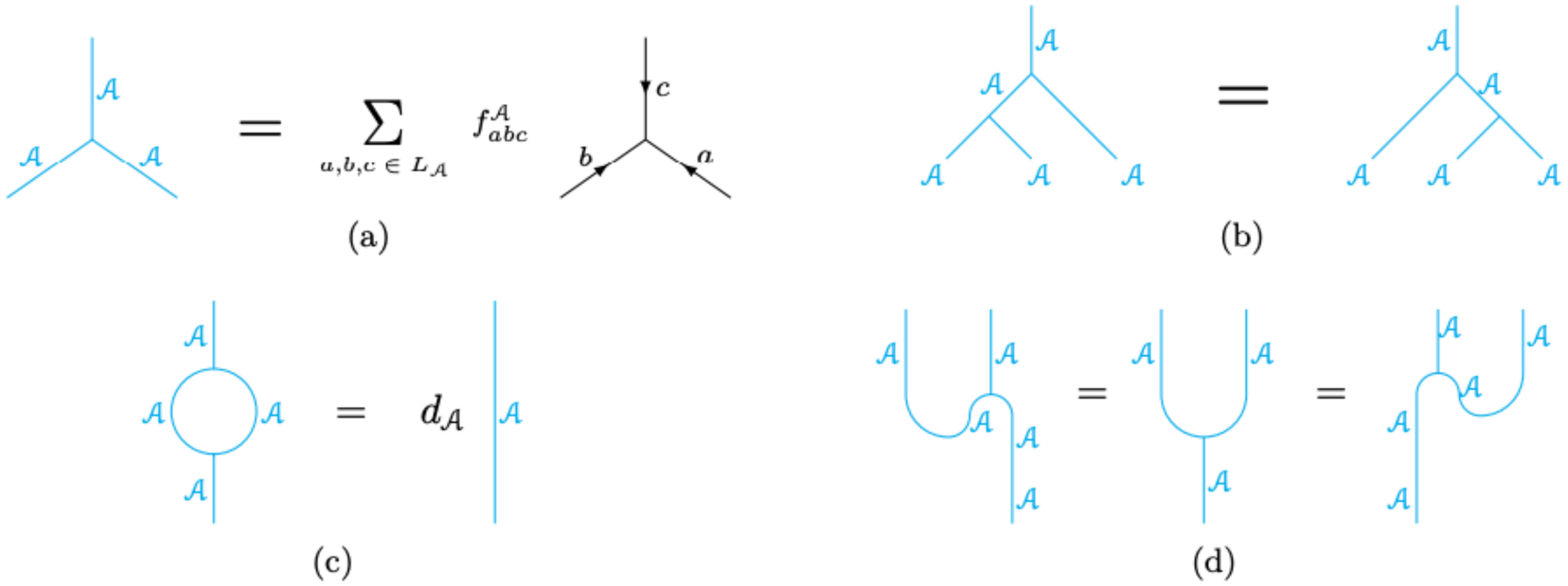
Two anyons with non-trivial braiding cannot condense together. CFT is an equilibrium point in this competition!

Ansatz of Unit cell

Module and Frobenius algebra

Fuchs, Runkel, Schweigert 2000s ; Hu, Wan, Wu 1706.00650;

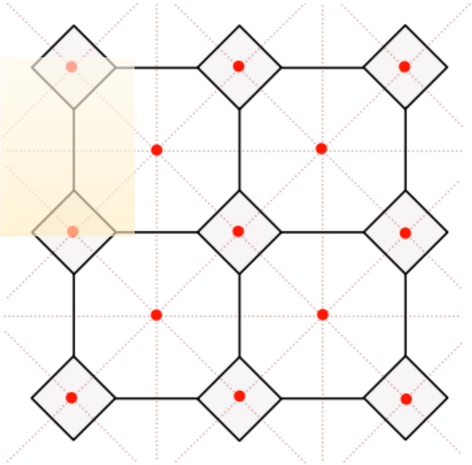
- A Frobenius algebra satisfies the following conditions:



Each Frobenius algebra describes a *Lagrangian algebra* which is a maximal set of anyons that condense. When a maximal set of anyons are condensed — this produces a trivial state. Each such trivial state is a topological boundary condition of the 3D TQFT.

Ansatz of Unit cell

Module and Frobenius algebra



Consider a octagon-square lattice. (Inspired by the Ising model)

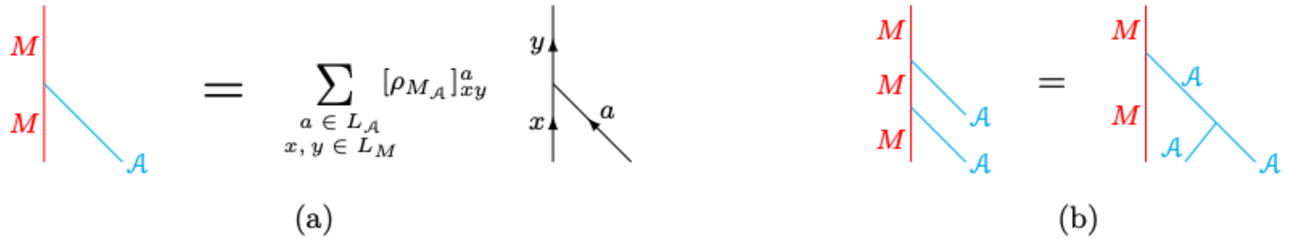
We consider a unit cell in this lattice as highlighted above. We want to define a state corresponding to pouring the condensate into this cell, while it is well “isolated” from the other cells.

This state is a topological boundary condition of the TQFT!

This is given by:

$$\langle \; \mathcal{A} \;\; M_{\mathcal{A}} \; | = \sum_{a \in L_{\mathcal{A}}} \sum_{x,y,u,v \in L_{M_{\mathcal{A}}}} \left\langle \begin{matrix} x & & y \\ & a & \\ u & & v \end{matrix} \right| \, [\rho_{M_{\mathcal{A}}}]^{a}_{xy} ([\rho_{M_{\mathcal{A}}}]^{a}_{uv})^{*}.$$

Here, M is a module of A. It satisfies:

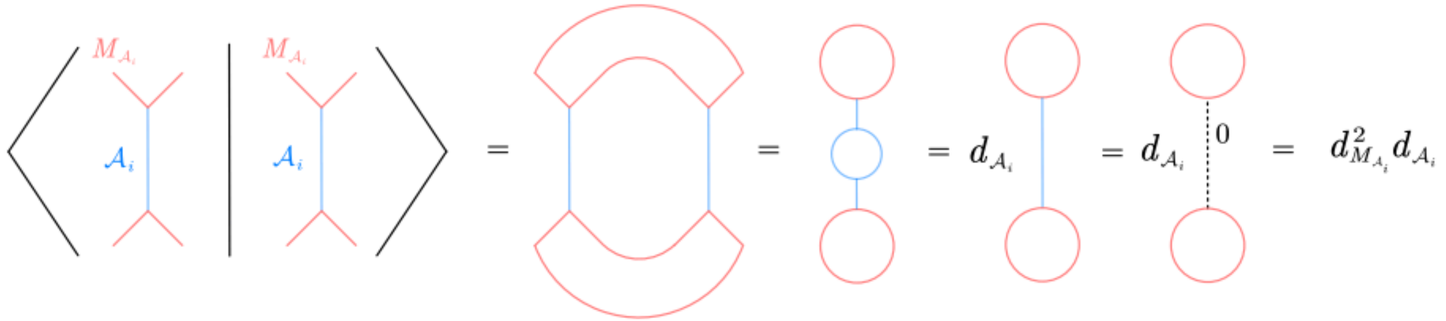


Ansatz of Unit cell

Module and Frobenius algebra

- Normalising a Frobenius algebra

$$N^2_{(\mathcal{A}_i, M_{\mathcal{A}_i})} =$$

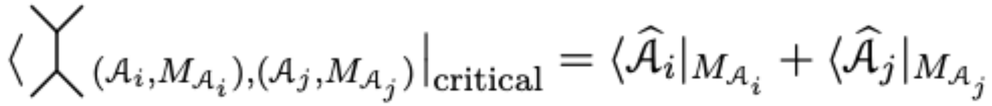


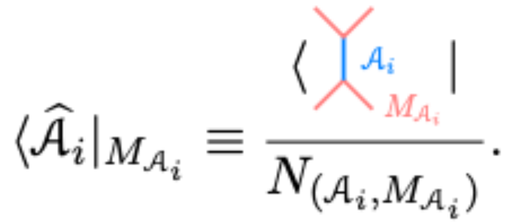
$$d_{M_{\mathcal{A}_i}} \equiv \sum_{c \in M_{\mathcal{A}_i}} m_c \, d_c, \qquad d_{\mathcal{A}_i} \equiv \sum_{a \in \mathcal{A}_i} n_a \, d_a.$$

Ansatz of Unit cell

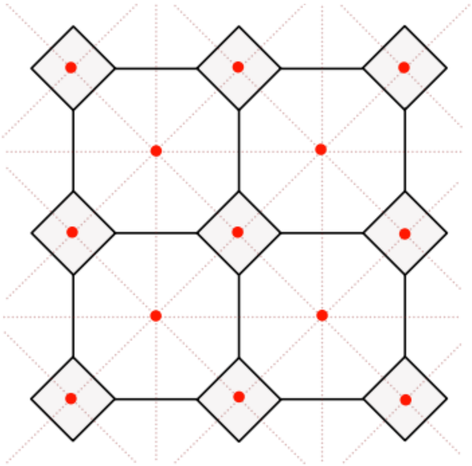
Module and Frobenius algebra

Interpolating between condensates:





Here, we require that Ai and Aj share the module objects in M, even if the module function may not be the same.
This is crucial to preserving the isolation of the condensate between unit cells in the full lattice.



Ansatz of Unit cell

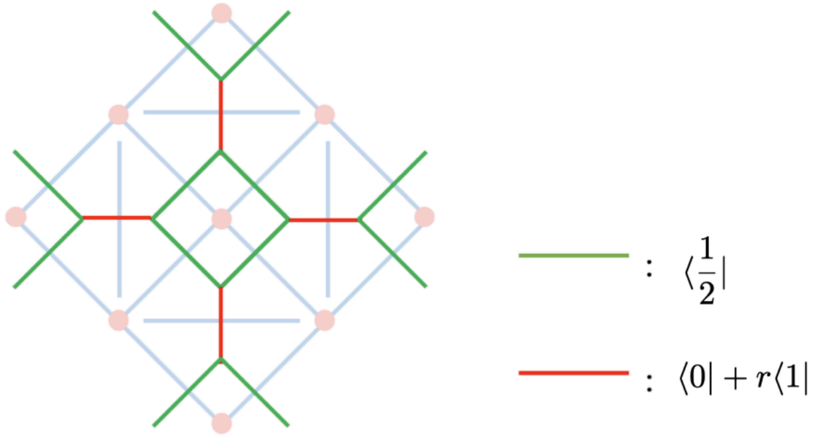
Module and Frobenius algebra

Example:
Going back to
the A-series

Interpolating between condensates:

$$\langle \; \rangle_{(\mathcal{A}_i,M_{\mathcal{A}_i}),(\mathcal{A}_j,M_{\mathcal{A}_j})}\Big|_{\text{critical}} = \langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} + \langle\widehat{\mathcal{A}}_j|_{M_{\mathcal{A}_j}}$$

$$\langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} \equiv \frac{\langle \; \rangle_{\mathcal{A}_i,M_{\mathcal{A}_i}}}{N_{(\mathcal{A}_i,M_{\mathcal{A}_i})}}.$$



For all k we have the following Frobenius algebra:

$$A_0 = 0$$

$$A_1 = 0 \oplus 1$$

$$\mathcal{M} = \frac{1}{2}$$

$$r_c = \sqrt{2\cos(2\pi/(k+2)) + 1}/(2\cos(\pi/(k+2) + 1)$$

Why does it work? — why is it a second order phase transition?

Key: cut down on degrees of freedom in RG space

- For chosen M such that it is shared by all the interpolating algebras (for simplicity, including module function), the RG attraction basins would be reduced to these algebras.

number of sharing A = n_M

  - This happens where $\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k$

Number of couplings in a unit cell:

$$D_{\text{unit}}(\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k) = N_{\mathcal{A}_k}. \qquad D_c = D_{\text{unit}} - 1$$

This is a sufficient but not *necessary* condition!

All phase transitions in a phase diagram are forced to be second order if

$$n_M > D_c.$$

i.e. all directions correspond to moving towards an incompatible phase.
No directions describe degeneracy between two phases.

Why does it work? — why is it a second order phase transition?

Key: cut down on degrees of freedom in RG space

$$\langle Y_{(\mathcal{A}_i,M_{\mathcal{A}_i}),(\mathcal{A}_j,M_{\mathcal{A}_j})}|_{\text{critical}} = \langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} + \langle\widehat{\mathcal{A}}_j|_{M_{\mathcal{A}_j}}$$

This expression reproduces the correct phase transition points in all the infinite set of examples below.

- Ak series examples : nM = 2, Dc = 1 $n_M > D_c.$ The entire series is second order phase transitions !

- N-state Potts model ($A_e = 0 \qquad A_m = 0 \oplus \cdots N-1 \qquad M = 0 \oplus \cdots N-1$)
  - N=2 nM = 2, Dc = 1 => 2nd order = Ising
  - N=3 nM = 2, Dc = 2 => 2nd order = 3 state Potts — seems to be an accident?
  - N=5 nM = 2, Dc= 4 => (weak) first order

Beyond N>5 well known that they are all first order transitions.

Phase Diagrams and Phase Boundaries

3 phase competition: Ashkin-Teller model from A5

phase boundaries and tri-critical point

- The module: $M = 1$ $A_0 = 0 \subset A_1 = 0 \oplus 4 \subset A_3 = 0 \oplus 4 \oplus 2$

Note that this choice of module kills $A_2 = 0 \oplus 2$

nM = 3, Dc = 2 => All phase transitions are second order.

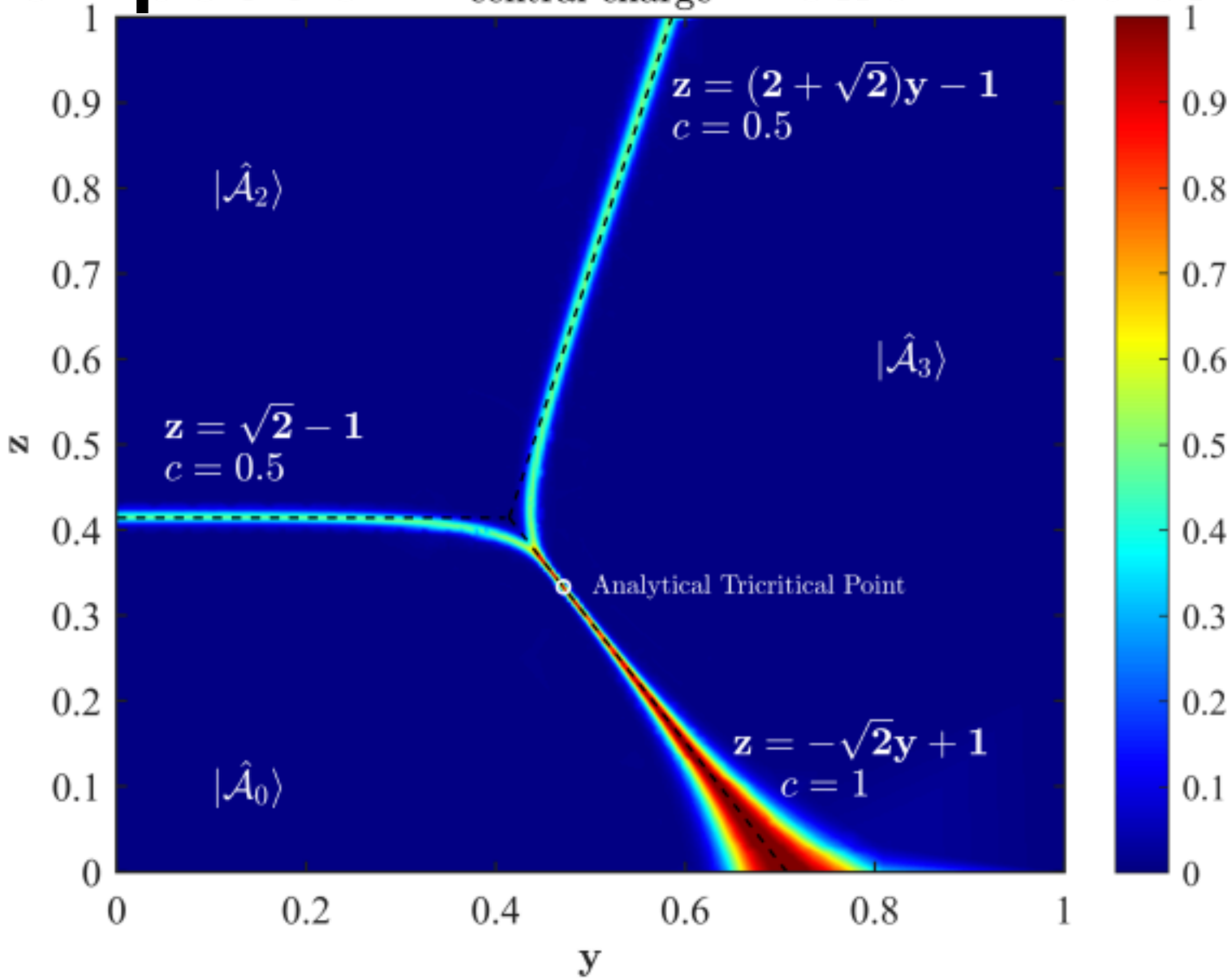
3 phase competition: Ashkin-Teller model from A5

phase bou

- The module

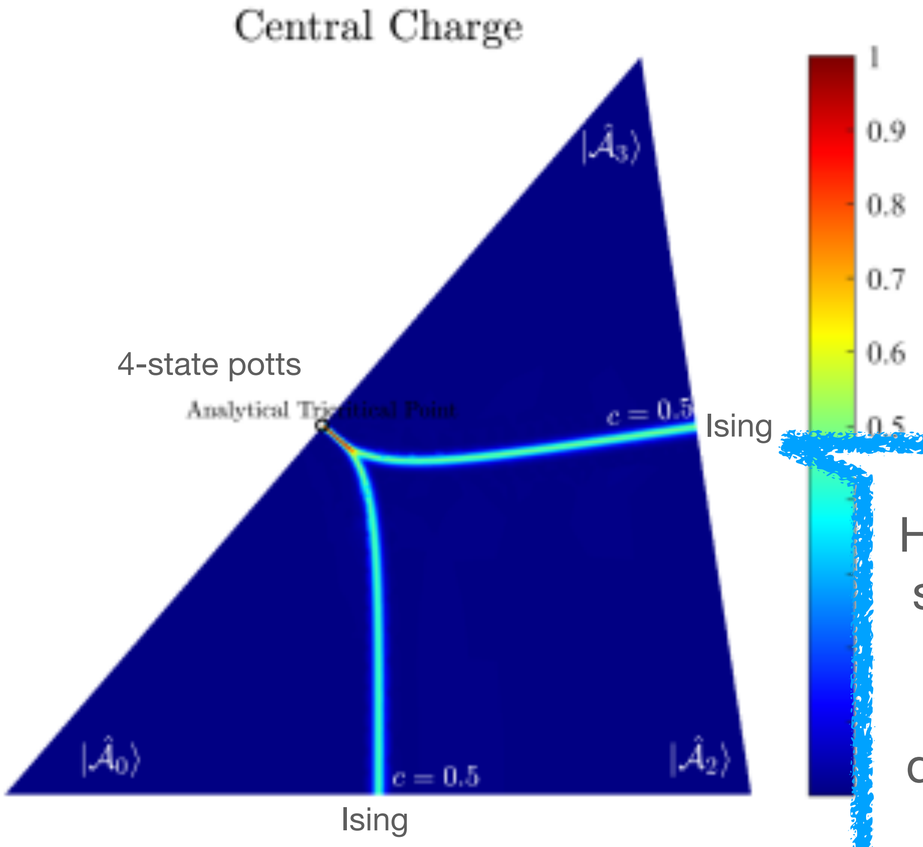
Note that this

nM = 3, Dc =
second order.



3 phase competition: Ashkin-Teller model from A5

phase boundaries and tri-critical point



Phase boundaries:

$$\langle C_{ij}| = \frac{1}{2}(\langle \hat{\mathcal{A}}_i| + \langle \hat{\mathcal{A}}_j|)$$

$$\langle C_{ij}(p)| = \langle C_{ij}| + p\left(\langle \hat{\mathcal{A}}_k| - \frac{\langle \hat{\mathcal{A}}_k|B_{ij}\rangle}{\langle B_{ij}|B_{ij}\rangle}\langle B_{ij}|\right)$$

How do we understand which CFT would show up here? Particularly they seem to preserve different amount of (non)-invertible symmetries. How do we determine what is the *minimal* amount of symmetries preserved?

Symmetries preserved — refined condensation tree

Refined condensation tree

Refining the ``Hasse diagram"

Chatterjee, Wen 22; Bhardwaj, Pajer, Schafer-Nemaki, Warman 24;

- Each node corresponds to a topological order obtained from partial anyon condensation of the parent phase - one draws a line connecting the child and parent phase

- From the perspective of the ``input-category" C, each node correspond to a sub-category K of C

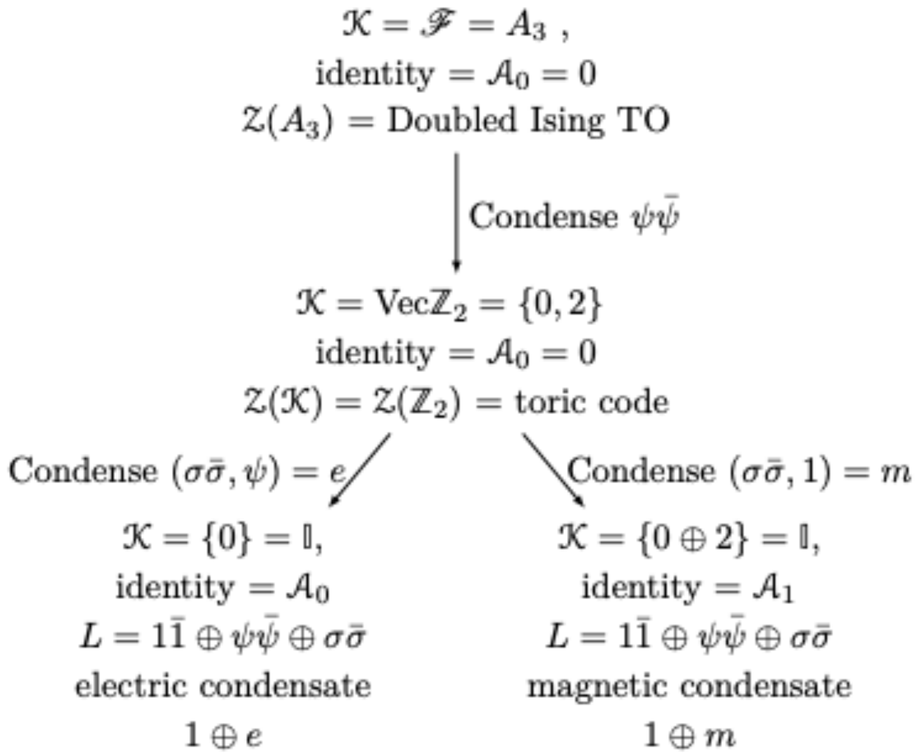
- Grow the tree from the top node corresponding to K = C (no condensation). The bottom nodes of the diagram are ``maximal" condensation corresponding to Lagrangian algebra L —now K becomes trivial with only 1 object, this object being a Frobenius algebra A .

Refined condensation tree

- Several Frobenius algebra A corresponds to the ``same" L
- But these L are actually different microscopically ! Different A does correspond to different boundary conditions. Two different A's correspond to the ``same" L can have non-trivial phase transitions between them.
- Our tree contains all these different A's that may correspond to the same L.

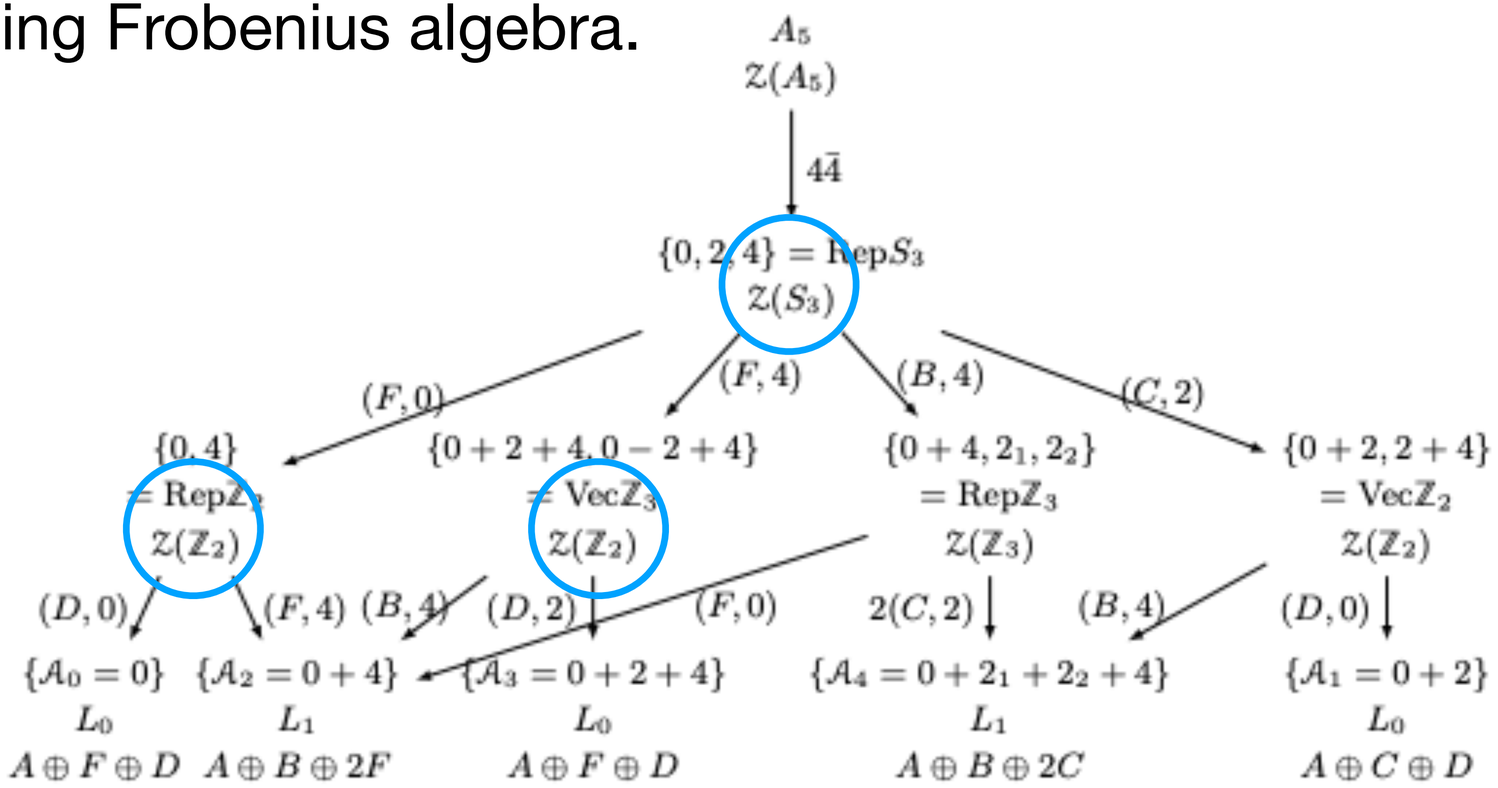
Refined condensation tree

- Examples: Ising



Refined condensation tree

- Examples: A5
  The minimal symmetry is determined by the first shared parent between to competing Frobenius algebra.



Novel Examples - Haagerup TQFT

Haagerup Model

$$1, \quad \alpha, \quad \alpha^2, \quad \rho, \quad \alpha\rho, \quad \alpha^2\rho,$$

The fusion rules are

1	α	α^2	ρ	$\alpha\rho$	$\alpha^2\rho$
α	α^2	1	$\alpha\rho$	$\alpha^2\rho$	ρ
α^2	1	α	$\alpha^2\rho$	ρ	$\alpha\rho$
ρ	$\alpha^2\rho$	$\alpha\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$
$\alpha\rho$	ρ	$\alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$
$\alpha^2\rho$	$\alpha\rho$	ρ	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$

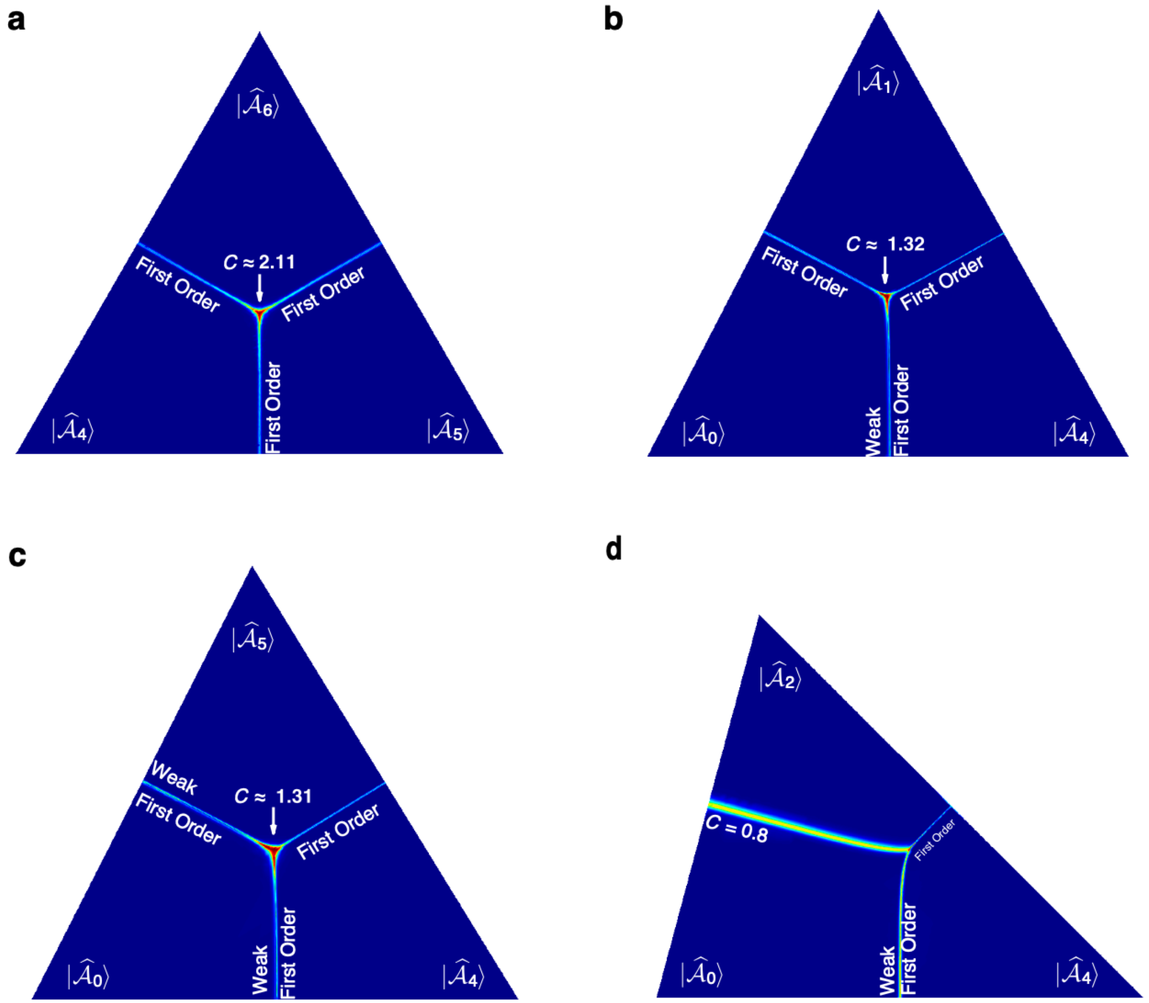
$$d_1 = d_\alpha = d_{\alpha^2} = 1, \qquad d_\rho = d_{\alpha\rho} = d_{\alpha^2\rho} = \frac{3+\sqrt{13}}{2}.$$

$$L_0 = 1 \oplus \pi_1 \oplus 2\pi_2$$
$$L_1 = 1 \oplus \pi_1 \oplus 2\sigma_1$$
$$L_2 = 1 \oplus \pi_1 \oplus \pi_2 \oplus \sigma_1.$$

$$\mathcal{A}_0 = 1$$
$$\mathcal{A}_1 = \rho \otimes \mathcal{A}_0 \otimes \rho = 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$$
$$\mathcal{A}_2 = 1 \oplus \alpha \oplus \alpha^2$$
$$\mathcal{A}_3 = \rho \otimes \mathcal{A}_2 \otimes \rho = (1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho) \otimes (1 \oplus \alpha \oplus \alpha^2).$$
$$\mathcal{A}_4 = 1 \oplus \rho \oplus \alpha\rho,$$
$$\mathcal{A}_5 = \alpha \otimes \mathcal{A}_4 \otimes \alpha^2 = 1 \oplus \alpha\rho \oplus \alpha^2\rho,$$
$$\mathcal{A}_6 = \alpha^2 \otimes \mathcal{A}_4 \otimes \alpha = 1 \oplus \rho \oplus \alpha^2\rho.$$

$$L_0 : \ \mathcal{A}_0, \mathcal{A}_1$$
$$L_1 : \ \mathcal{A}_2, \mathcal{A}_3$$
$$L_2 : \ \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6.$$

Haagerup Model



Haagerup Model - summary of results

H_3	$\mathcal{A}_0,\ \mathcal{A}_1$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	3	First	—	—	—
H_3	$\mathcal{A}_0,\ \mathcal{A}_2$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	2	Cont.	$\mathcal{Z}(S_3)$	0.801	0.013
H_3	$\mathcal{A}_0,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	2	Weak	—	1.3	—
H_3	$\mathcal{A}_1,\ \mathcal{A}_2$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	5	First	—	—	
H_3	$\mathcal{A}_1,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	5	First	—	—	
H_3	$\mathcal{A}_2,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	4	First	—	—	
H_3	$\mathcal{A}_{(4,5,6)},\ \mathcal{A}_{(5,6,4)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	4	First	—	—	
H_3	$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	5	Cont.	$\mathcal{Z}(H_3)$	0.793	0.014
H_3	$\mathcal{A}_0,\ \mathcal{A}_2,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	4	Cont.	$\mathcal{Z}(H_3)$	0.8047	0.0077
H_3	$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	5	Cont.	$\mathcal{Z}(H_3)$	1.3154	0.0065
H_3	$\mathcal{A}_0,\ \mathcal{A}_{(4,5,6)},\ \mathcal{A}_{(5,6,4)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	4	Cont.	$\mathcal{Z}(H_3)$	1.3141	0.0042
H_3	$\mathcal{A}_1,\ \mathcal{A}_{(4,5,6)},\ \mathcal{A}_{(5,6,4)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	7	First	—	—	—
H_3	$\mathcal{A}_2,\ \mathcal{A}_{(4,5,6)},\ \mathcal{A}_{(5,6,4)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	6	First	—	—	—
H_3	$\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_{(4,5,6)}$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	7	First	—	—	—
H_3	$\mathcal{A}_4,\ \mathcal{A}_5,\ \mathcal{A}_6$	$\rho \oplus \alpha\rho \oplus \alpha^2\rho$	6	Cont.	$\mathcal{Z}(H_3)$	2.106	0.022

Symmetry preserving RG

Generalised Symmetry Preserving RG and Their Fixed Point

A lattice integrable model can also be written in this form:
R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete;
D. Aasen, P. Fendley, and R. S. K. Mong

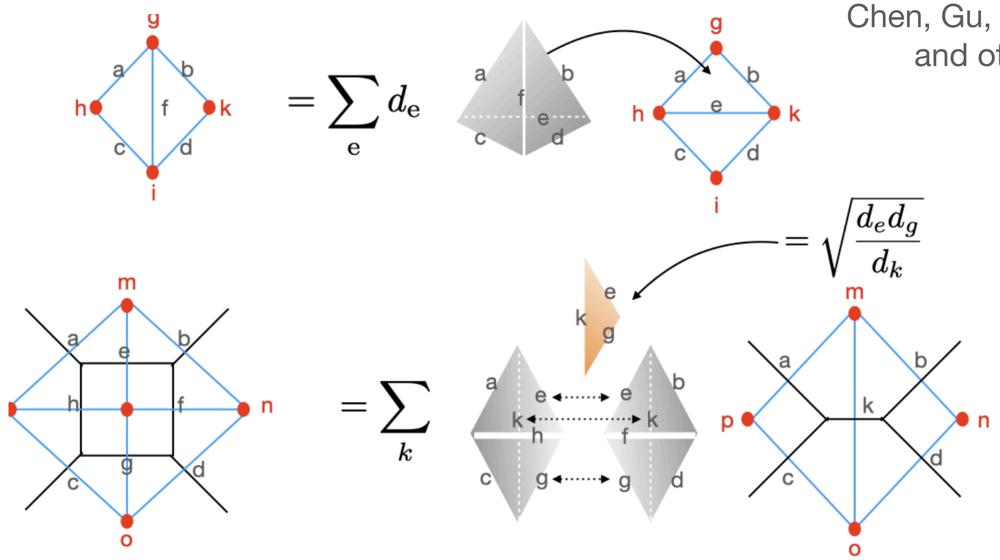
$$\langle \Omega_N | \Psi_a^{LW} \rangle$$

$$|\Psi\rangle_\Lambda = U|\Psi\rangle_{\sqrt{2}\Lambda}$$

$$\langle \Omega_N | FF | \Psi_{ka}^{LW} \rangle = \langle \Omega_{N-1} | \Psi_{ka}^{LW} \rangle$$

$$\sqrt{2}_\Lambda \langle \Omega | =_\Lambda \langle \Omega | U$$

RG of topological order
Chen, Gu, Wen 2010;
and others..



- symmetric topological phase = topological eigenstate of RG operator from Frobenius algebra.
- symmetric CFT is an infinite bond dimension eigenstate of the RG operator from conformal blocks

Gong Cheng, Lin Chen, Zheng-Cheng Gu
2311.18005
Phys.Rev.X 15 (2025) 1, 011073

Lin Chen, Kaixin Ji, Haochen Zhang, Ce Shen, Ruoshui Wang
2210.12127 Physical Review X 14 (4), 041033

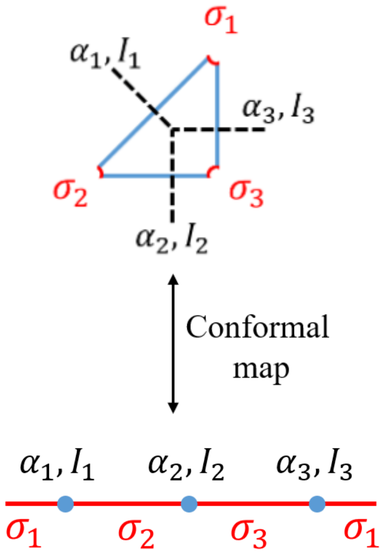
CFT is a symmetric RG fixed point

Cut 2D CFT path-integral into pieces

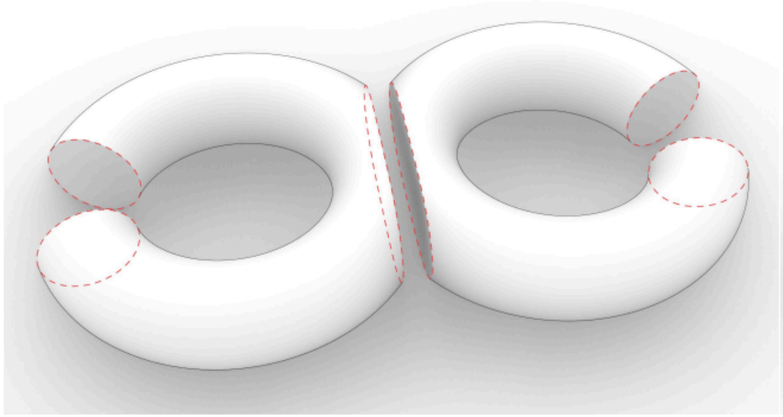
Gabriel Wong, LYH 2020
Lin Chen, Kaixin Ji, Haochen Zhang, Ce Shen, Ruoshui Wang, LYH 2022
Brehm, Runkel 2022

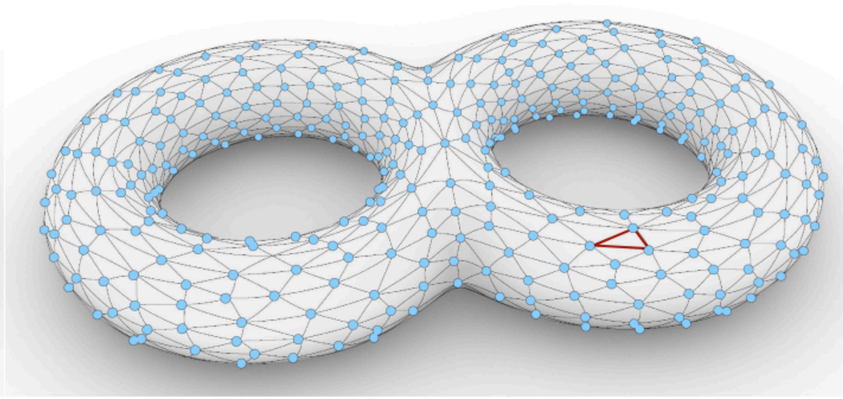
- Cut 2D manifold into triangles.
- Each triangle = three-point function of boundary-changing operators.
- Glue triangles: summing over states on the edge (primaries + descendants).
- Leaving holes (radius R) at each vertex, labeled by conformal boundary conditions.

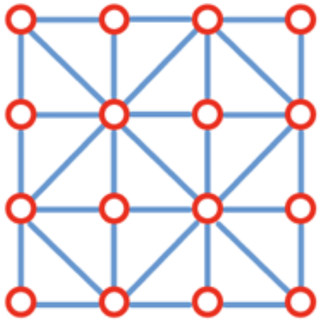
$$\mathcal{T}^{\sigma_1\sigma_2\sigma_3}_{(\alpha_1,I_1)(\alpha_2,I_2)(\alpha_3,I_3)}(\triangle, R) =$$



$$\mathcal{Z}(R, \sigma_a) = \sum_{\{\alpha_i\}} \sum_{\{I_i\}} \prod_{\triangle} \mathcal{T}^{\sigma_a\sigma_b\sigma_c}_{(\alpha_i,I_i)(\alpha_j,I_j)(\alpha_k,I_k)}(\triangle, R)$$

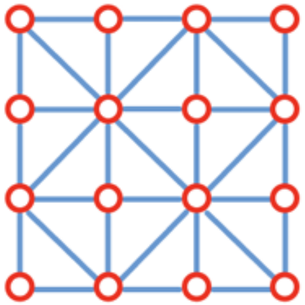






Cut a CFT into triangles

$$\mathcal{Z}(R,\sigma_a) = \sum_{\{\alpha_i\}}\sum_{\{I_i\}}\prod_{\triangle}\mathcal{T}^{\sigma_a\sigma_b\sigma_c}_{(\alpha_i,I_i)(\alpha_j,I_j)(\alpha_k,I_k)}(\triangle,R)$$



- Shrink the holes.

Shrinkable Boundary Condition =
"Entanglement Brane Boundary Condition" in RCFT
Wong, LYH 2020,
Brehm, Runkel 2022

Vacuum Ishibashi state $|e\rangle\rangle = \sum_a \omega_{\sigma_a}|\sigma_a\rangle$

"Quantum dimensions"

Shrinkable boundary designed to preserve **all** the CFT generalised symmetries.

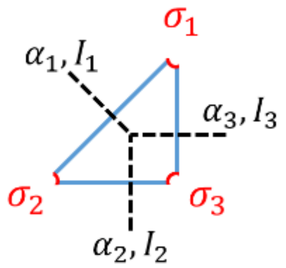
$$\mathcal{Z}(R) = \sum_{\{\sigma_a\}}\prod_v \omega_{\sigma_a}\sum_{\{\alpha_i\}}\sum_{\{I_i\}}\prod_{\triangle}\mathcal{T}^{\sigma_a\sigma_b\sigma_c}_{(\alpha_i,I_i)(\alpha_j,I_j)(\alpha_k,I_k)}(\triangle,R)\ .$$

$$Z_{\mathrm{CFT}} = \lim_{R\to 0}\prod_v e^{-c/6\ln R}\mathcal{Z}(R)$$

Emergence of 3D Bulk from BCFT OPE

Lin Chen, Kaixin Ji, Haochen Zhang, Ce Shen, Ruoshui Wang, LYH 2022

$$\mathcal{Z}(R) = \sum_{\{\sigma_a\}} \prod_v \omega_{\sigma_a} \sum_{\{\alpha_i\}} \sum_{\{I_i\}} \prod_{\triangle} \mathcal{T}^{\sigma_a\sigma_b\sigma_c}_{(\alpha_i,I_i)(\alpha_j,I_j)(\alpha_k,I_k)}(\triangle, R) \ .$$



$$\equiv \tilde{C}^{\sigma_3\sigma_1\sigma_2}_{\alpha_1\alpha_2\alpha_3} \tilde{\gamma}^{\alpha_1\alpha_2\alpha_3}_{I_1I_2I_3}$$

related to “strange correlator” in lattice spin models
Bal, Williamson, Vanhove, Bultinck, Haegeman, Verstraete 2018;
Aasen, Mong, Fendley 2016; 2020;

Rewrite Z(R) as an overlap of two 3D states

$$\mathcal{Z}(R) = \langle \Omega(R) | \Psi \rangle = \sum_{\{\alpha_i\}} \Omega(\{\alpha_i\})(R) \Psi(\{\alpha_i\})$$

Collection of BCFT OPE (= 6j symbols= 3D tetrahedra)

Fuchs, Runkel, Schweigert 2002-2006

$$\Psi(\{\alpha_i\}) = \sum_{\{\sigma_a\}} \prod_v \omega_{\sigma_a} \prod_{\triangle} \tilde{C}^{\sigma_a\sigma_b\sigma_c}_{\alpha_i\alpha_j\alpha_k}$$

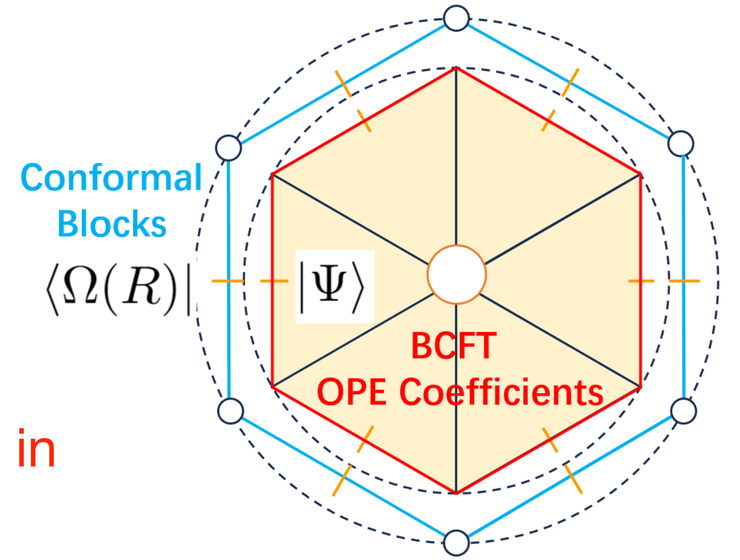
Ground state of a **3D** TQFT

Turaev, Viro 1992;
Levin, Wen 2004

Collection of conformal blocks

$$\Omega(\{\alpha_i\})(R) = \sum_{\{I_i\}} \prod_{\triangle} \tilde{\gamma}^{\alpha_i\alpha_j\alpha_k}_{I_iI_jI_k}(R)$$

3D graph independence in the bulk triangulation



Low dimensional AdS/CFT?

Liouville CFT - Closing Holes

Lin Chen, LYH, Yikun Jiang, Bingxin Lao 2024

$$\Phi_{\alpha}^{\sigma_1\sigma_2}(x)$$

$$\sigma \equiv Q/2 + iP_\sigma, \qquad \alpha \equiv Q/2 + iP_\alpha$$

A self-consistent collection of conformal boundary conditions

Fateev, Zamolodchikov, Zamolodchikov 2000; Teschner 2000;

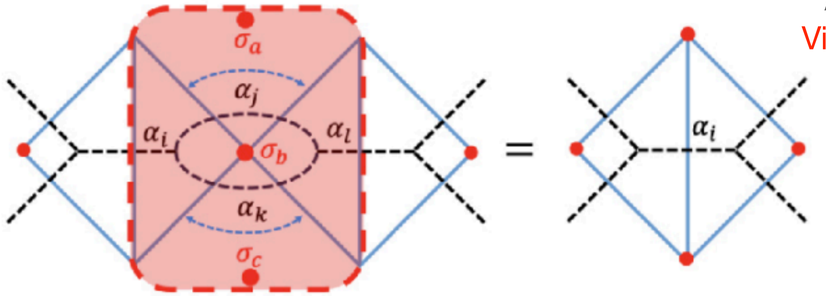
$$C_{\alpha_1,\alpha_2,\alpha_3}^{\sigma_3,\sigma_1,\sigma_2} = (\mu(P_{\alpha_1})\mu(P_{\alpha_2})\mu(P_{\alpha_3}))^{1/4} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{Bmatrix}_b .$$

Ponsot, Teschner 2001; Chen, LYH, Jiang, Lao 2024;

$$Q = b + b^{-1} \qquad c = 1 + 6Q^2$$

$$\mathcal{U}_q(SL(2,\mathbb{R})) \qquad q = e^{\pi i b^2}$$

Also Crossing Kernel of Virasoro conformal blocks



$$\mu(P) = 4\sqrt{2}\sinh(2\pi P b)\sinh\left(\frac{2\pi P}{b}\right).$$

Weight the hole with the Plancherel measure (Cardy density of states)

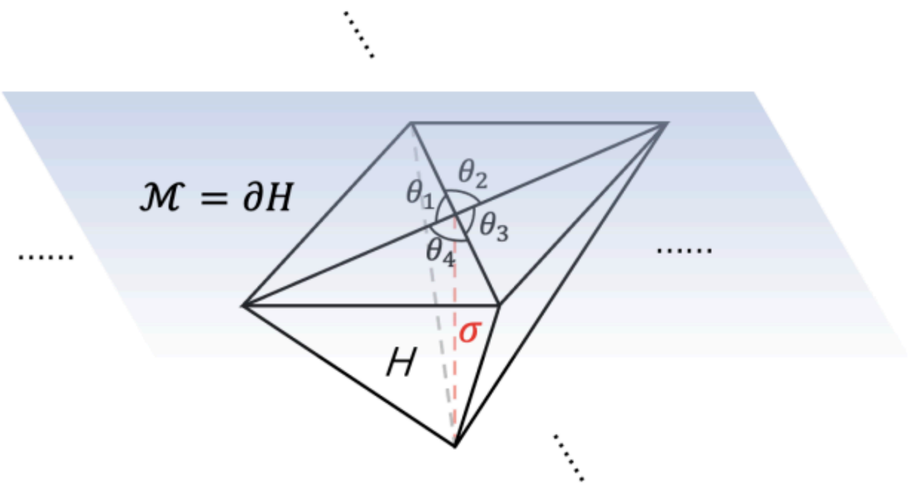
Reading off the bulk

Lin Chen, LYH, Yikun Jiang, Bingxin Lao 2024

$$Z_{\text{Liouville}}(R) = \langle \Omega(R) | \Psi \rangle$$

$$\Psi(\{\alpha_i\}) = \prod_v \int_0^\infty dP_{\sigma_a} \mu(P_{\sigma_a}) \prod_e \sqrt{\mu(P_{\alpha_i})} \prod_\triangle \begin{Bmatrix} \alpha_i & \alpha_j & \alpha_k \\ \sigma_c & \sigma_a & \sigma_b \end{Bmatrix}_b$$

$$\Omega(\{\alpha_i\})(R) = \prod_e \sum_{\{I_i\}} \prod_\triangle \left(\tilde{\gamma}^{\alpha_i \alpha_j \alpha_k}_{I_i I_j I_k}(R) \right)$$



Reading off the bulk in the large c limit

$$P = \frac{l}{2\pi b}$$

$$\left\{ \begin{matrix} \frac{Q}{2}+i\frac{l_4}{2\pi b} & \frac{Q}{2}+i\frac{l_5}{2\pi b} & \frac{Q}{2}+i\frac{l_6}{2\pi b} \\ \frac{Q}{2}+i\frac{l_1}{2\pi b} & \frac{Q}{2}+i\frac{l_2}{2\pi b} & \frac{Q}{2}+i\frac{l_3}{2\pi b} \end{matrix} \right\}_b \underset{b\to 0}{\to} \exp\left(-\frac{\mathrm{Vol}(T\{l_i\})+\sum_i l_i\theta_i/2}{\pi b^2}\right),$$

Kashaev 1997;
Teschner, Vartanov 2012;
Collier, Eberhardt, Zhang 2023;
Chen, LYH, Jiang, Lao 2024;

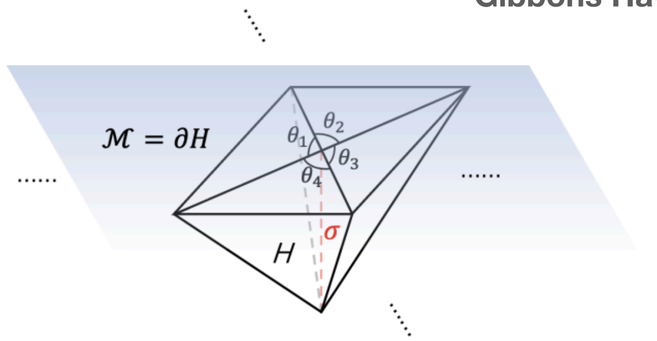
Large c limit reduces to on-shell Einstein-Hilbert action on hyperbolic tetrahedron.

$$S_{EH} = -\frac{1}{16\pi G_N}\int_H d^3x\sqrt{g}(R-2\Lambda) = \frac{V(H)}{4\pi G_N} = \frac{V(H)}{\pi b^2},$$

$$\sum_i \frac{\theta_i l_i}{2\pi b^2} = \sum_i \frac{\theta_i l_i}{8\pi G_N} = \frac{1}{8\pi G_N}\sum_i \int_{\Gamma_i} \theta_i \sqrt{h},$$

Gibbons Hawking/Hayward Term

BCFT primary labels ⟺ Bulk geodesic lengths



Summary and Outlook

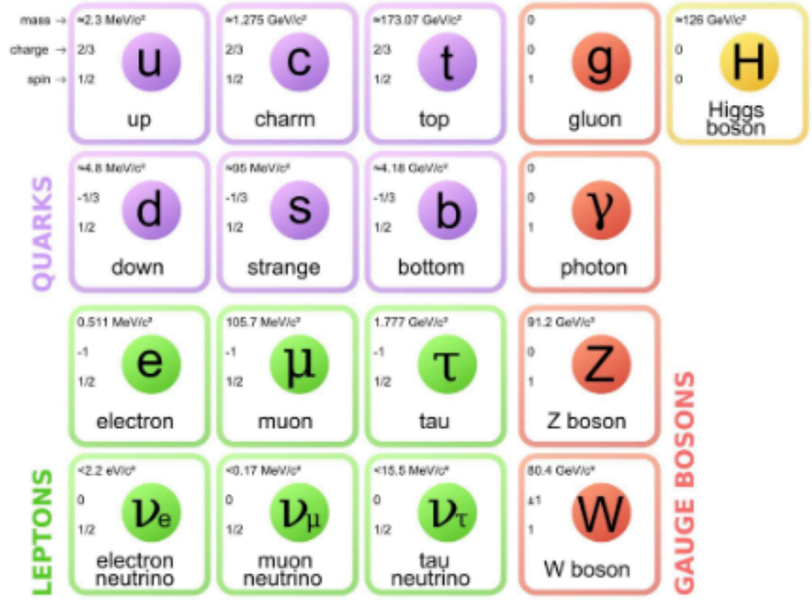
- Topological holographic principle — connects the discrete and the continuous world
- AdS/CFT seems to coincide with the topological holographic principle, at least in low dimensions!
-

Future: Interacting Field Theory

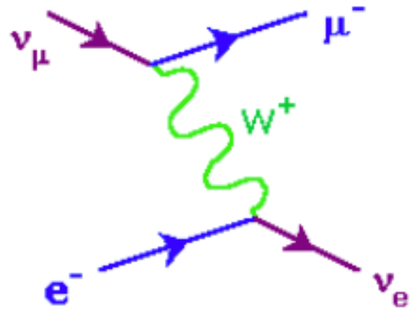
- Quantum Field Theory seems to be the correct language to describe the natural forces, particularly, the standard model

pictures courtesy Cern

Quantum Field Theory and the Electroweak Standard Model - Arbuzov, A.B. - arXiv:1801.05670



Particle content of the Standard Model. Courtesy to Wikipedia: 'Standard Model of Elementary Particles' by MissMJ --- Own work by uploader, PBS NOVA, Fermilab, Office of Science, United States Department of Energy, Particle Data Group.



Feynman diagram for electron-neutrin scattering with W boson exchange.

But the success is mostly
in the high energy limit, when the theory approaches free theory— where perturbation based on Feynman diagrams have been extremely successful.

Thank you!